Assigning Controllability and Observability Gramians in Feedback Control

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Almost all robustness properties of linear systems can be related directly to properties of the controllability or observability Gramian. This paper shows how to assign the controllability or observability Gramian of the closed-loop system. All Gramian controllers are also parameterized in terms of model data in a state-space representation.

I. Introduction

CONSIDER now a linear system

$$\dot{x} = Ax + Dw$$

where $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$, and the controllability Gramian

$$X \triangleq \int_0^\infty e^{At}DWD^*e^{A^*t}\,dt$$

For any $W>0$ it is well known that $X>0$ if and only if A,D is an observable pair. For stable A, X satisfies

$$O = XA^* + AX + DWD^*$$

We wish to characterize the set of all X that may be assigned to the system by any controller. Such parameterization requires two steps: 1) deriving the necessary and sufficient conditions for the existence of some controller to assign X, and 2) finding the set of all controllers that assign X.

The covariance assignment problem was first defined in Ref. 1 and solved for the state feedback case (see also Refs. 2 and 3). Subsequently, these results have been extended to the dynamic controller of any order for continuous systems, and all controllers that assign a specified state covariance to a continuous-time system are derived in Ref. 4. This paper extends the ideas of Ref. 4 to the deterministic problem of assigning prescribed matrix values to the closed-loop controllability and observability Gramians.

The Gramian controllers are parameterized in terms of a matrix having physical significance (the controllability, observability Gramians). This gives a multiobjective flavor to the controller capability, because the $n(n+1)/2$ parameter values are assigned to the closed-loop system.

Almost all robustness properties of linear systems can be related directly to properties of the controllability or observability Gramian (see Refs. 5 and 6). For example, the linear system

$$\dot{x} = (A + \Delta A)x$$

with stable A remains stable for all ΔA satisfying

$$\bar{\sigma}[\Delta A] < (\bar{\sigma}[K])^{-1}$$

where K is the observability Gramian associated with the matrix pair $(A, \sqrt{2I})$. That is, K satisfies

$$0 = KA + A^*K + 2I$$

Bounds[6] on ΔA can also be written in terms of a controllability Gramian X

$$\bar{\sigma}[\Delta A] < \frac{\underline{\sigma}(DD^*)^{1/2}}{2\bar{\sigma}[X(DD^*)^{-1/2}]}$$

where X is the controllability Gramian associated with the matrix pair A, D. That is, X satisfies

$$0 = XA^* + AX + DD^*$$

Reference 6 demonstrates that bounds based on the controllability Gramian X can provide a less conservative bound than that based on the observability Gramian K, but not always. Hence, there is motivation to develop a control design method that can assign either the controllability or the observability Gramian, since such a result would allow control designs that guarantee a specified degree of robustness.

The Gramians are also related to the L_2 norm of the system output. For example, the stable system

$$\dot{x} = Ax + Dw, \qquad y = Cx, \qquad w \in \mathbb{R}^l$$

has the output L_2 norm

$$\int_0^\infty y^*Qy\,dt = \mathrm{tr}XC^*QC = \mathrm{tr}KDWD^*$$

where

$$0 = KA + A^*K + C^*QC$$

$$0 = XA^* + AX + DWD^*$$

if $w(t)$ is an impulse with strength $\sqrt{W}$. Similar results are available for the multi-input case in Ref. 2. Hence, when X or K is prespecified, a control design would yield a guaranteed

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cost controller in the sense that the output L_2 norm to an impulsive input would take on a prespecified value.

This paper is organized as follows. Section II discusses the problem statement. Section III states the assignability conditions, and the resulting controllers are parameterized. Section IV provides the dual results for observability Gramian assignment. Section V gives some numerical examples. Section VI offers some conclusions.

II. Problem Statement

Consider the following problem. Given any linear system of order n_x

$$\dot{x}_p = A_p x_p + B_p u + D_p w \tag{1a}$$

$$z = M_p x_p \tag{1b}$$

where $x_p \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ the input, $w \in \mathbb{R}^{n_w}$ the disturbances, and $z \in \mathbb{R}^{n_z}$ the output measurement, find all linear controllers of order n_c

$$\dot{x}_c = A_c x_c + B_c z \tag{2a}$$

$$u = C_c x_c + G_c z \tag{2b}$$

that will stabilize the closed-loop system and assign the Gramian value X to the system. X is partitioned as

$$X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^* & X_c \end{bmatrix} \tag{3}$$

We call this the controllability Gramian assignment problem.

By defining the matrices

$$A \triangleq \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_p & 0 \\ 0 & I_{n_c} \end{bmatrix}, \quad M \triangleq \begin{bmatrix} M_p & o \\ 0 & I_{n_c} \end{bmatrix}$$

$$G \triangleq \begin{bmatrix} G_c & C_c \\ B_c & A_c \end{bmatrix}, \quad D \triangleq \begin{bmatrix} D_p \\ 0 \end{bmatrix}$$

where I_{n_c} denotes an $n_c \times n_c$ identity matrix, it is possible to write the closed-loop system of Eqs. (1) and (2) in the form

$$\dot{x} = (A + BGM)x + Dw \tag{4}$$

Then, as is well known, the matrix X of Eq. (4) defined in Eq. (3) satisfies

$$0 = X(A + BGM)^* + (A + BGM)X + DWD^* \tag{5}$$

where

$$W > 0$$

Hence, the Gramian assignment problem is to find G such that Eq. (5) is satisfied for a specified $X > 0$ and $A + BGM$ is asymptotically stable.

Definition: Let Eqs. (1-3) describe a linear dynamic system. If there exists a set of matrices (A_c, B_c, C_c, G_c) such that $X = \bar{X}$, then $\bar{X}$ is called an assignable controllability Gramian.

III. Main Results

The main results of this paper are the existence conditions for a controller of order n_c that will assign X to the closed-loop system and the characterization of all controllers that can assign X to the system.

Theorem 1: A specified controllability Gramian $X > 0$ is assignable by some G if and only if

$$(I - B_p B_p^+) Q_p (I - B_p B_p^+) = 0 \tag{6}$$

$$(I - M_p^+ M_p) \bar{Q}_p (I - M_p^+ M_p) = 0 \tag{7}$$

and either

$$M^+ M X^{-1} Q (I - BB^+)(I - \Gamma^+ \Gamma) = 0 \tag{8}$$

or

$$(I - LL^+)(I - M^+ M) X^{-1} Q BB^+ = 0 \tag{9}$$

are satisfied, where $^+$ denotes the Moore-Penrose inverse, and

$$Q_p \triangleq X_p A_p^* + A_p X_p + D_p W D_p^* \tag{10a}$$

$$\bar{Q}_p \triangleq \bar{X}_p^{-1} (\bar{X}_p A_p^* + A_p \bar{X}_p + D_p W D_p^*) \bar{X}_p^{-1} \tag{10b}$$

$$\bar{X}_p \triangleq X_p - X_{pc} X_c^{-1} X_{pc}^* (>0) \tag{10c}$$

$$\Gamma \triangleq M^+ M X (I - BB^+) \tag{10d}$$

$$L \triangleq (I - M^+ M) X^{-1} BB^+ \tag{10e}$$

$$Q \triangleq XA^* + AX + DWD^* \tag{10f}$$

Proof: See Appendix A.

The conditions shown in Theorem 1 are equivalent to those obtained in Ref. 4 because both of them are a necessary and sufficient condition for the problem considered here. However, conditions (7-9) are considerably simpler. The notable importance of Theorem 1 is for the representation to be given without any singular value decomposition. Hence, it can be said that the condition shown in the theorem is more explicit (i.e., closed form) than the previous one. These features are also applicable to the following theorem, which gives the characterization of Gramian controllers.

Theorem 2: Suppose X is assignable. Then all controllers that assign X to the system (4) are given by

$$G = -\tfrac{1}{2} B^+ X (2I - M^+ M) \bar{Q} M^+ + \tfrac{1}{2} B^+ X M^+ M (\Phi^* - \Phi) M^+ + B^+ X M^+ M (I - \Gamma\Gamma^+) \hat{S} (I - \Gamma\Gamma^+) M^+ + \hat{Z} - B^+ B \hat{Z} M M^+ \tag{11}$$

where

$$\bar{Q} \triangleq X^{-1}(XA^* + AX + DWD^*) X^{-1} = X^{-1} Q X^{-1} \tag{12a}$$

$$\Phi \triangleq 2\bar{Q} X (I - BB^+) \Gamma^+ + \Gamma\Gamma^+ \bar{Q} [I - X(I - BB^+)\Gamma^+] \tag{12b}$$

or, equivalently,

$$G = -\tfrac{1}{2} B^+ Q (2I - BB^+) X^{-1} M^+ + \tfrac{1}{2} B^+ (\psi^* - \psi) BB^+ \times X^{-1} M^+ + B^+ (I - L^+ L) \hat{S} (I - L^+ L) BB^+ X^{-1} M^+ + \hat{Z} - B^+ B \hat{Z} M M^+ \tag{13}$$

where

$$\psi \triangleq 2L^+ (I - M^+ M) X^{-1} Q + [I - L^+ (I - M^+ M) X^{-1}] Q L^+ L \tag{14}$$

In Eqs. (11) and (13), $\hat{S}$ is an arbitrary skew-Hermitian matrix and $\hat{Z}$ is an arbitrary matrix of appropriate dimension.

Proof: See Appendix B.

Remark 1: Since $B(\hat{Z} - B^+ B \hat{Z} M M^+) M = 0$, the choice of the free parameter matrix $\hat{Z}$ does not influence output performance of the closed-loop system, nor the system matrix A_c of the controller, because the submatrix of

$\hat{Z} - B^+ B\hat{Z}MM^+$ corresponding to A_c is zero. These facts are trivial in the case that both B_p and M_p are of full rank since $B^+B = I$ and $MM^+ = I$.

Remark 2: The skew-Hermitian matrix $\hat{S}$ is $(n_x + n_c) \times (n_x + n_c)$; however, the degree of freedom in $\hat{S}$ parameters is less than the degree that of an $\gamma \times \gamma$ skew-Hermitian matrix, where $\gamma = \min\ \{\text{rank}(I - \Gamma\Gamma^+),\ \text{rank}(I - L^+L)\}$, as discussed in Ref. 4.

A sufficient condition for the closed-loop system (4) to be stable is as follows.

Lemma 1: Suppose that (A_p, B_p) is a controllable pair and Eq. (5) holds for some $X > 0$. Then $A + BGM$ is asymptotically stable for any given G in Eq. (11) or (13) if matrices

$$\begin{bmatrix} A_p - \lambda I & D_p \\ M_p & 0 \end{bmatrix} \tag{15}$$

$$[A_c - \lambda I \quad B_c] \tag{16}$$

both have full row rank for all λ on $j\omega$ axis.

Proof: See Appendix C.

This lemma is more general than that given in Ref. 4. By setting $n_c = 0$, we can readily specialize Theorems 1 and 2 to the case of static controllers. In this case, $A \to A_p$, $B \to B_p$, $D \to D_p$, $M \to M_p$, $G \to G_c$, $x \to x_p$ and matrices A_c, B_c, M_c, and I_{n_c} vanish. Especially, putting $M = M_p = I$ implies $M^+ = M_p^+ = I$ and $L = L^+ = 0$, leading to the following result for the case of state feedback.

Corollary to Theorems 1 and 2: The state controllability Gramian $X_p > 0$ is assignable to the system (4) by a state feedback controller if and only if

$$(I - B_pB_p^+)Q_p(I - B_pB_p^+) = 0 \tag{17}$$

and the set of all state feedback gains G that assign X_p to the system (4) is given by

$$\begin{aligned} G = G_c = &- \tfrac{1}{2}B_p^+ Q_p(2I - B_pB_p^+)X_p^{-1} \\ &+ B_p^+ \hat{S} B_pB_p^+ X_p^{-1} + (I - B_p^+B_p)\hat{Z} \end{aligned} \tag{18}$$

where $\hat{S}$ is an arbitrary skew-Hermitian matrix and $\hat{Z}$ is an arbitrary matrix.

IV. Observability Gramian Assignment

The following gives the corresponding theory for the assignment of the observability Gramian. The proofs follow in a straightforward way by substituting the dual for system (A_p, B_p, M_p, X_p).

Theorem 3: Let the pair (A_p, M_p) be observable. There exists a G such that

$$0 = K(A + BGM) + (A + BGM)^*K + M^*M \tag{19}$$

for some specified $K > 0$

$$K = \begin{bmatrix} K_p & K_{pc} \\ K_{pc}^* & K_c \end{bmatrix}$$

if and only if

$$(I - M_p^+M_p)R_p(I - M_p^+M_p) = 0 \tag{20}$$

$$(I - B_pB_p^+)\bar{R}_p(I - B_pB_p^+) = 0 \tag{21}$$

and either

$$BB^+K^{-1}R(I - M^+M)(I - \Gamma_K^+\Gamma_K) = 0 \tag{22}$$

or

$$(I - L_KL_K^+)(I - BB^+)K^{-1}RM^+M = 0 \tag{23}$$

are satisfied, where

$$R_p \triangleq K_pA_p + A_p^*K_p + M_p^*M_p \tag{24a}$$

$$\bar{R}_p \triangleq \bar{K}_p^{-1}(\bar{K}_pA_p + A_p^*\bar{K}_p + M_p^*M_p)\bar{K}_p^{-1} \tag{24b}$$

$$\bar{K}_p \triangleq K_p - K_{pc}K_c^{-1}K_{pc}^* \ (>0) \tag{24c}$$

$$\Gamma_K \triangleq BB^+K(I - M^+M) \tag{24d}$$

$$L_K \triangleq (I - BB^+)K^{-1}M^+M \tag{24e}$$

$$R \triangleq KA + A^*K + M^*M \tag{24f}$$

The set of all G is given by

$$\begin{aligned} G = &- \tfrac{1}{2}B^+\bar{R}(2I - BB^+)KM^+ \\ &+ \tfrac{1}{2}B^+(\Phi_K - \Phi_K^*)BB^+KM^+ \\ &+ B^+(I - \Gamma_K\Gamma_K^+)\hat{S}(I - \Gamma_K\Gamma_K^+)BB^+KM^+ \\ &+ \hat{Z} - B^+B\hat{Z}MM^+ \end{aligned} \tag{25}$$

where

$$\bar{R} \triangleq K^{-1}RK^{-1} \tag{26a}$$

$$\begin{aligned} \Phi_K \triangleq\ & 2\bar{R}K(I - M^+M)\Gamma_K^+ \\ &+ \Gamma_K\Gamma_K^+\bar{R}[I - K(I - M^+M)\Gamma_K^+] \end{aligned} \tag{26b}$$

or, equivalently,

$$\begin{aligned} G = &- \tfrac{1}{2}B^+K^{-1}(2I - M^+M)RM^+ \\ &+ \tfrac{1}{2}B^+K^{-1}M^+M(\psi_K - \psi_K^*)M^+ \\ &+ B^+K^{-1}M^+M(I - L_K^+L_K)\hat{S}(I - L_K^+L_K)M^+ \\ &+ \hat{Z} - B^+B\hat{Z}MM^+ \end{aligned} \tag{27}$$

where

$$\begin{aligned} \psi_K \triangleq\ & 2L_K^+(I - BB^+)K^{-1}R \\ &+ [I - L_K^+(I - BB^+)K^{-1}]RL_K^+L_K \end{aligned} \tag{28}$$

where $\hat{S}$ is an arbitrary skew-Hermitian matrix and $\hat{Z}$ is an arbitrary matrix of approximate dimension.

We conclude from these results that a specified $K > 0$ is assignable as an observability Gramian to the closed-loop system if and only if the conditions (20–21), and either (22) or (23), are satisfied.

From this theorem, we can immediately get the following corollary for the state feedback case.

Corollary of Theorem 3: Suppose that the pair (A_p, M_p) is observable. A specified $K_p > 0$ can be assigned as the observability Gramian of the closed system by a state feedback controller if and only if

$$(I - B_pB_p^+)\bar{R}_p(I - B_pB_p^+) = 0 \tag{29}$$

All state feedback gains for Gramian assignment are parameterized by

$$\begin{aligned} G = &-\frac{1}{2}B_p^+\bar{R}_p(2I - B_pB_p^+)K_p + B_p^+\hat{S}B_pB_p^+K_p \\ &+ (I - B_p^+B_p)\hat{Z} \end{aligned} \tag{30}$$

V. Examples

Example 1: Consider the pitch motion of a rigid spacecraft

$$J\ddot{\theta} = u, \qquad x_p = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$z_p = \theta, \qquad J = 1$$

If the disturbance w represents an error in computing angular rate, the state-space model is given by

$$A_p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$M_p = [1 \; 0] \tag{31}$$

The assignable controllability Gramian (for a first-order controller) may be written as

$$X = \begin{bmatrix} x_{11} & x_{12} & x_a \\ x_{12} & x_{22} & x_b \\ x_a & x_b & x_c \end{bmatrix} = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^* & X_c \end{bmatrix}$$

The assignability condition (6) in Theorem 1 gives $x_{12} = -\frac{1}{2}$ and condition (7) gives

$$\frac{1}{2} + \frac{x_a x_b}{x_c} = 0 = -2 \det \bar{X}_p$$

Condition (8) is necessarily satisfied because

$$\Gamma = \begin{bmatrix} x_{11} \\ 0 \\ x_a \end{bmatrix} [1 \; 0 \; 0], \qquad I - \Gamma\Gamma^+ = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

which yields $(I - BB^+)(I - \Gamma^+\Gamma) = 0$ independently of the values x_{11} and x_a. Hence, all assignable controllability Gramians are given as

$$X = \begin{bmatrix} x_{11} & -\frac{1}{2} & x_a \\ -\frac{1}{2} & x_{22} & x_b \\ x_a & x_b & x_c \end{bmatrix} \quad (>0)$$

where, from condition (7),

$$2x_a x_b + x_c = 0$$

or

$$2x_c x_{11} x_{22} - 2x_b^2 x_{11} - 2x_a^2 x_{22} - x_a x_b = 0$$

Example 2: For the same system (31), consider assigning the observability Gramian,

$$K = \begin{bmatrix} k_{11} & k_{12} & k_a \\ k_{12} & k_{22} & k_b \\ k_a & k_b & k_c \end{bmatrix} = \begin{bmatrix} K_p & K_{pc} \\ K_{pc}^* & K_c \end{bmatrix}$$

From condition (20) in Theorem 3, we get $k_{12} = 0$, and from condition (21) we get $(k_c k_{22} - k_b)^2 + 2k_a k_b k_c \det \bar{K}_p = 0$. In this case, condition (22) is always satisfied, and the assignable observability Gramian is given as

$$K = \begin{bmatrix} k_{11} & 0 & k_a \\ 0 & k_{22} & k_b \\ k_a & k_b & k_c \end{bmatrix}$$

where k_{11}, k_{22} and k_c are positive and $k_b^2 < k_1 k_{22}$ for positive definiteness and the remaining freedom is characterized by

$$(k_c k_{22} - k_b^2)^2 + 2k_a k_b k_{11} (k_c k_{22} - k_b^2) - 2k_a^3 k_b k_{22} = 0$$

VI. Conclusions

All controllers of any specified order are parameterized; they assign specified values to the controllability or observability Gramian. These parameterizations are given in closed form; that is, they are explicit in terms of model data in a state-space representation.

Appendix A: Proof of Theorem 1

We need the singular value decomposition (SVD) of some matrices to give this proof. By defining the SVD of B_p as

$$B_p = [U_{b1} \quad U_{b2}] \begin{bmatrix} \Sigma_b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{b1}^* \\ V_{b2}^* \end{bmatrix}$$

we can get the SVD of B as follows

$$B = \begin{bmatrix} B_p & 0 \\ 0 & I_{n_c} \end{bmatrix}$$

$$= \begin{bmatrix} U_{b1} & 0 & U_{b2} \\ 0 & I_{n_c} & 0 \end{bmatrix} \begin{bmatrix} \Sigma_b & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{b1}^* & 0 \\ 0 & I_{n_c} \\ V_{b2}^* & 0 \end{bmatrix}$$

$$\triangleq [U_{B1} \quad U_{B2}] \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{B1}^* \\ V_{B2}^* \end{bmatrix}$$

$$= U_{B1} \Sigma_B V_{B1}^*$$

Similarly, define the SVD of M_p as

$$M_p = [U_{m1} \quad U_{m2}] \begin{bmatrix} \Sigma_m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{m1}^* \\ V_{m2}^* \end{bmatrix}$$

and we can write

$$M = \begin{bmatrix} M_p & 0 \\ 0 & I_{n_c} \end{bmatrix}$$

$$= \begin{bmatrix} U_{m1} & 0 & U_{m2} \\ 0 & I_{n_c} & 0 \end{bmatrix} \begin{bmatrix} \Sigma_m & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{m1}^* & 0 \\ 0 & I_{n_c} \\ V_{m2}^* & 0 \end{bmatrix}$$

$$\triangleq [U_{M1} \quad U_{M2}] \begin{bmatrix} \Sigma_M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{M1}^* \\ V_{M2}^* \end{bmatrix}$$

$$= U_{M1} \Sigma_M V_{M1}^*$$

For convenience, we shall use the following matrices from now on.

$$\Delta_1 \triangleq U_{B1} U_{B1}^* = BB^+, \qquad \Delta_2 \triangleq U_{B2} U_{B2}^* = I - BB^+$$

$$\nabla_1 \triangleq V_{M1} V_{M1}^* = M^+ M, \qquad \nabla_2 \triangleq V_{M2} V_{M2}^* = I - M^+ M$$

The proof starts from showing the following lemma:

Lemma A1: A specified $X > 0$ is assignable if and only if

$$(I - B_p B_p^+) Q_p (I - B_p B_p^+) = 0 \tag{A1}$$

$$(I - M_p^+ M_p) \bar{Q}_p (I - M_p^+ M_p) = 0 \tag{A2}$$

and there exists an $S_{11} (= -S_{11}^*)$ and an $\bar{S}_{11} (= -\bar{S}_{11}^*)$ such that

$$U_B \begin{bmatrix} S_{11} & Q_{12} \\ -Q_{12}^* & 0 \end{bmatrix} U_B^* = X V_M \begin{bmatrix} \bar{S}_{11} & -\bar{Q}_{21}^* \\ \bar{Q}_{21} & 0 \end{bmatrix} V_M^* X \tag{A3}$$

where

$$Q_{12} \triangleq U_{B1}^* Q U_{B2}, \qquad \bar{Q}_{21} \triangleq V_{M2}^* X^{-1} Q X^{-1} V_{M1}$$

Proof of Lemma A1: A given $X>0$ is assignable if there exists a G satisfying Eq. (5), that is,

$$(BGMX)^T + (BGMX) + Q = 0$$

which requires G to satisfy

$$BGMX = -\tfrac{1}{2}(Q+S), \qquad S = -S^* \tag{A4}$$

for some skew-Hermitian matrix S. Equation (A4) has a solution GMX at least. This gives a necessary condition such that

$$(I - BB^+)(Q+S) = 0 \tag{A5}$$

Similarly, since Eq. (A4) has a solution BG, we get another necessary condition such that

$$(Q+S)X^{-1}(K - M^+M) = 0 \tag{A6}$$

By multiplying U_B^* from the left and U_B from the right, Eq. (A5) becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q_{11}+S_{11} & Q_{12}+S_{12} \\ Q_{21}+S_{21} & Q_{22}+S_{22} \end{bmatrix} = 0$$

where $Q_{ij} \triangleq U_{Bi}^* Q U_{Bj}$ and $S_{ij} \triangleq U_{Bi}^* S U_{Bj}$ $(i = 1,2;\ j = 1,2)$. Since $Q_{22} = Q_{22}^*$, $Q_{22} + S_{22} = 0$ means $Q_{22} = 0$, that is,

$$U_B^*(I - BB^+)U_B U_B^* Q U_B U_B^* (I - BB^+) U_B = 0$$

This is just equivalent to

$$\begin{bmatrix} I - B_pB_p^+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_p & A_pX_{pc} \\ X_{pc}^*A_p^* & 0 \end{bmatrix} \begin{bmatrix} I - B_pB_p^+ & 0 \\ 0 & 0 \end{bmatrix} = 0$$

which yields Eq. (A1). $Q_{21} + S_{21} = 0$ with $Q_{22} = S_{22} = 0$ gives the form of S as follows.

$$S = U_B \begin{bmatrix} S_{11} & Q_{12} \\ -Q_{21}^* & 0 \end{bmatrix} U_B^*, \qquad S_{11} = -S_{11}^* \tag{A7}$$

In a similar way, from Eq. (A6), we get Eq. (A2) and the form for S to satisfy as follows.

$$S = XV_M \begin{bmatrix} \bar{S}_{11} & -\bar{Q}_{21}^* \\ \bar{Q}_{21} & 0 \end{bmatrix} V_M^* X, \qquad \bar{S}_{11} = -\bar{S}_{11}^*$$

which is the same as in Eq. (A7). This fact gives the third condition.

To prove sufficiency, suppose that Eqs. (A5) and (A6) are simultaneously satisfied for some $S(=-S^*)$. Then identify

$$(I - BB^+)(Q+S)X^{-1}(I - M^+M)X - (I - BB^+)(Q+S) - (Q+S)X^{-1}(I - M^+M)X = 0$$

holds and this is equivalent to

$$BB^+(Q+S)X^{-1}M^+MX = Q + S$$

which is no other than the condition for solvability of Eq. (A4). (QED)

From now on we prove that the condition (A3) in Lemma A1 can be equivalently exchanged for condition (8) or (9) in Theorem 1.

Equation (A3) can be rewritten as

$$[U_{B1} \quad XV_{M1}] \begin{bmatrix} S_{11} & 0 \\ 0 & -\bar{S}_{11} \end{bmatrix} \begin{bmatrix} U_{B1}^* \\ V_{M1}^* X \end{bmatrix} = S_Q \tag{A8}$$

where

$$S_Q \triangleq -QX^{-1}\nabla_2 X + X\nabla_2 X^{-1} Q - Q\Delta_2 + \Delta_2 Q (= -S_Q^*)$$

which can be derived by using the relation $Q_{22} = U_{B2}^* Q U_{B2} = 0$ and $\bar{Q}_{22} = V_{M2}^* X^{-1} Q X^{-1} V_{M2} = 0$. It should be noted that $\nabla_2 X^{-1} S_Q \Delta_2 = 0$. By premultiplying U_B^* and postmultiplying U_B to Eq. (A8), we get

$$\begin{bmatrix} I & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{X}^* \tilde{S} \tilde{X} \begin{bmatrix} I & 0 \\ I & 0 \\ 0 & I \end{bmatrix} = U_B^* S_Q U_B \tag{A9}$$

where

$$\tilde{X} = \begin{bmatrix} I & 0 \\ 0 & V_M^* X U_B \end{bmatrix}, \qquad \tilde{S} = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & -\bar{S}_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equation (A9) has a skew-Hermitian solution $\tilde{S}$ and all of the solutions are represented by

$$\tilde{S} = (\tilde{X}^*)^{-1} \left\{ \begin{bmatrix} I & I & 0 \\ 0 & 0 & I \end{bmatrix}^+ U_B^* S_Q U_B \begin{bmatrix} I & 0 \\ I & 0 \\ 0 & I \end{bmatrix}^+ + Z - \begin{bmatrix} I & I & 0 \\ 0 & 0 & I \end{bmatrix}^+ \begin{bmatrix} I & I & 0 \\ 0 & 0 & I \end{bmatrix} Z \begin{bmatrix} I & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} I & 0 \\ I & 0 \\ 0 & I \end{bmatrix}^+ \right\} \tilde{X}^{-1}$$

$$= (\tilde{X}^*)^{-1} \left\{ \begin{bmatrix} \tfrac{1}{2}I & 0 \\ \tfrac{1}{2}I & 0 \\ 0 & I \end{bmatrix} U_B^* S_Q U_B \begin{bmatrix} \tfrac{1}{2}I & \tfrac{1}{2}I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} (1/\sqrt{2})I & (-1/\sqrt{2})I & 0 \\ (1/\sqrt{2})I & (1/\sqrt{2})I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & Z_1 & 0 \\ -Z_1^* & Z_2 & Z_3 \\ 0 & -Z_3^* & 0 \end{bmatrix} \times \begin{bmatrix} (1/\sqrt{2})I & (1/\sqrt{2})I & 0 \\ (-1/\sqrt{2})I & (1/\sqrt{2})I & 0 \\ 0 & 0 & I \end{bmatrix} \right\} \tilde{X}^{-1} \tag{A10}$$

where Z is an arbitrary skew-Hermitian matrix and Z_2 is also an arbitrary skew-Hermitian matrix such that the 1-2th, 1-3th, 3-2th and 3-3th blocks corresponding to the portion of $\tilde{S}$ are zero matrices. This requires the following restriction to Z_{12}, Z_{22}, and Z_{23}:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \{\text{right side of Eq. (A10)}\} \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} = 0$$

which can be written down as

$$[\sqrt{2}I \quad -I \quad 0]\tilde{Z} \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} = -\frac{1}{2} U_{B1}^* S_Q U_B \begin{bmatrix} \tfrac{1}{2}I & 0 \\ 0 & I \end{bmatrix} \tag{A11}$$

$$V_{M2}^*X^{-1}U_B\begin{bmatrix}0 & I & 0\\ 0 & 0 & I\end{bmatrix}\tilde{Z}\begin{bmatrix}0 & 0\\ I & 0\\ 0 & I\end{bmatrix}$$

$$= -V_{M2}^*X^{-1}U_B\begin{bmatrix}\tfrac{1}{2}I & 0\\ 0 & I\end{bmatrix}U_B^*S_QU_B\begin{bmatrix}\tfrac{1}{2}I & 0\\ 0 & I\end{bmatrix} \tag{A12}$$

where $\tilde{Z}$ is a skew-Hermitian matrix such as

$$\tilde{Z} = \begin{bmatrix}0 & (1/\sqrt{2})Z_1 & 0\\ (-1/\sqrt{2})Z_1^* & \tfrac{1}{2}(Z_1 - Z_1^* + Z_2) & (1/\sqrt{2})Z_3\\ 0 & (-1/\sqrt{2})Z_3^* & 0\end{bmatrix}$$

$$= \begin{bmatrix}I & 0 & 0\\ (1/\sqrt{2})I & (1/\sqrt{2})I & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}0 & Z_1 & 0\\ -Z_1^* & Z_2 & Z_3\\ 0 & -Z_3^* & 0\end{bmatrix}$$

$$\times\begin{bmatrix}I & (1/\sqrt{2})I & 0\\ 0 & (1/\sqrt{2})I & 0\\ 0 & 0 & I\end{bmatrix}$$

Only for convenience, we now put

$$\tilde{Z}_1 \triangleq \begin{bmatrix}0 & I & 0\\ 0 & 0 & I\end{bmatrix}\tilde{Z}\begin{bmatrix}I & 0\\ 0 & 0\\ 0 & I\end{bmatrix}(= -\tilde{Z}_1^*)$$

$$\tilde{S}_1 \triangleq \begin{bmatrix}\tfrac{1}{2}I & 0\\ 0 & I\end{bmatrix}U_B^*S_QU_B\begin{bmatrix}\tfrac{1}{2}I & 0\\ 0 & I\end{bmatrix}(= -\tilde{S}_1^*)$$

and rewrite Eq. (A12) equivalently as

$$V_{M2}^*X^{-1}U_B(\tilde{Z}_1 + \tilde{S}_1)U_B^*X^{-1} = 0 \tag{A13}$$

It can be seen by modifying the results in Ref. 6 that Eq. (A13) has a skew-Hermitian solution $\tilde{Z}_1 + \tilde{S}_1$ and the general solutions are given as

$$\tilde{Z}_1 + \tilde{S}_1 = U_B^*X\nabla_1S_Z\nabla_1XU_B$$

which yields

$$\begin{bmatrix}\tfrac{1}{2}(Z_1 - Z_1^* + Z_2) & (1/\sqrt{2})Z_3\\ (-1/\sqrt{2})Z_3^* & 0\end{bmatrix}$$

$$= -\tilde{S}_1 + U_B^*X\nabla_1S_Z\nabla_1XU_B \tag{A14}$$

where S_Z is an arbitrary skew-Hermitian matrix such that

$$U_{B2}^*X\nabla_1S_Z\nabla_1XU_{B2} = U_{B2}^*S_QU_{B2} \tag{A15}$$

from the reason why the 2-2th block of the right side must be zero. By substituting the value of $[\tfrac{1}{2}(Z_1 - Z_1^* + Z_2)\ (1/\sqrt{2})Z_3]$ in Eq. (A14) into Eq. (A11), we get

$$\sqrt{2}[Z_1\ 0] = -U_{B1}^*S_QU_B\begin{bmatrix}\tfrac{1}{2}I & 0\\ 0 & I\end{bmatrix}$$

$$+ U_{B1}^*X\nabla_1S_Z\nabla_1XU_B \tag{A16}$$

which requires that S_Z also must make the 1-2th block of the right side of Eq. (A16) zero, that is,

$$U_{B1}^*X\nabla_1S_Z\nabla_1XU_{B2} = U_{B1}^*S_QU_{B2} \tag{A17}$$

Hence, we can find some $\tilde{S}$ satisfying Eq. (A9) if and only if there exists some skew-Hermitian matrix S_Z such that both Eqs. (A15) and (A17) or, equivalently,

$$X\nabla_1S_Z\nabla_1XU_{B2} = S_QU_{B2} \tag{A18}$$

holds. Then, the $\tilde{S}$ is given from Eqs. (A10), (A14), and (A16) as

$$\tilde{S} = \begin{bmatrix}U_{B1}^*(S_Q - X\nabla_1S_Z\nabla_1X)U_{B1} & 0 & 0\\ 0 & V_{M1}^*S_ZV_{M1} & 0\\ 0 & 0 & 0\end{bmatrix} \tag{A19}$$

The solvability of Eq. (A18), which is equivalent to condition (8) in Theorem 1, is provided in the following Lemma.

Lemma A2: The linear equation (A18) has a skew-Hermitian solution S_Z 1) if and only if

$$\nabla_1X^{-1}Q\Delta_2(I - \Gamma^+\Gamma) = 0 \tag{A20}$$

in which case a general skew-Hermitian solution is

$$S_Z = \nabla_1X^{-1}S_Q\Delta_2\Gamma^+ + (\Gamma^+)^*\Delta_2S_QX^{-1}\nabla_1$$

$$-(\Gamma^+)^*\Delta_2S_Q\Delta_2\Gamma^+ + \nabla_2Z_s\Gamma^+ - (\nabla_2Z_s\Gamma^+)^*$$

$$+ (I - \Gamma\Gamma^+)S_s(I - \Gamma\Gamma^+) \tag{A21}$$

where Z_s is an arbitrary matrix and S_s is an arbitrary skew-Hermitian matrix, or, equivalently, 2) if and only if

$$(I - LL^+)\nabla_2X^{-1}Q\Delta_1 = 0 \tag{A22}$$

in which case a general skew-Hermitian solution is

$$S_Z = \nabla_1X^{-1}(S_Q + \Delta_1S_L\Delta_1)X^{-1}\nabla_1 + S_s - \nabla_1S_s\nabla_1 \tag{A23a}$$

$$S_L \triangleq -\Delta_1S_QX^{-1}\nabla_2L^+ - (L^+)^*\nabla_2X^{-1}S_Q\Delta_1$$

$$+ (L^+)^*\nabla_2X^{-1}S_QX^{-1}\nabla_2L^+ + \Delta_2Z_0L^+ - (\Delta_2Z_0L^+)^*$$

$$+ (I - LL^+)S_0(I - LL^+) \tag{A23b}$$

where S_s and S_0 are arbitrary skew-Hermitian matrices, and Z_0 is an arbitrary matrix.

Proof of Lemma A2: We first prove part 1. Premultiplying Eq. (A18) by a nonsingular matrix $V_M^*X^{-1}$ gives

$$\begin{bmatrix}V_{M1}^*S_Z\nabla_1XU_{B2}\\ 0\end{bmatrix} = \begin{bmatrix}V_{M1}^*X^{-1}S_QU_{B2}\\ V_{M2}^*X^{-1}S_QU_{B2}\end{bmatrix}$$

in which the second equation is always satisfied and the first one can be rewritten as

$$V_{M1}^*S_Z\nabla_1X[0\ U_{B2}] = V_{M1}^*X^{-1}S_Q[0\ U_{B2}]$$

Postmultiplying by a nonsingular matrix U_B^*, we get

$$V_{M1}^*S_Z\Gamma = V_{M1}^*X^{-1}S_Q\Delta_2 \tag{A24}$$

which is equivalent to Eq. (A18). A necessary condition for this equation to have a solution is

$$\nabla_1X^{-1}S_Q\Delta_2(I - \Gamma^+\Gamma) = 0 \tag{A25}$$

which yields Eq. (A20) by substituting S_Q defined in Eq. (A8).

To give a general skew-Hermitian solution to Eq. (A24), we first observe that the following skew-Hermitian matrix S_Z^s

satisfies Eq. (A24) being a special solution subject to Eq. (A25):

$$S_Z^s = \nabla_1 X^{-1} S_Q \Delta_2 \Gamma^+ + (\Gamma^+)^* \Delta_2 S_Q X^{-1} \nabla_1 - (\Gamma^+) \Delta_2 S_Q \Delta_2 \Gamma^+ \quad \text{(A26)}$$

in which the identity $\Delta_2 S_Q \Delta_2 = \Delta_2 X \nabla_1 X^{-1} S_Q \Delta_2$ can help to see. Now we may consider a solution to a homogeneous equation

$$V_{M1}^* S_Z^H \Gamma = 0 \quad \text{(A27)}$$

To be consistent, Eq. (A27) requires that $S_Z^H \Gamma$ should be of the form

$$S_Z^H \Gamma = \nabla_2 Z^H \quad \text{(A28)}$$

where Z^H is an arbitrary matrix. Moreover, in order that this equation has a skew-Hermitian solution S_Z^H, the matrix Z^H must satisfy $\nabla_2 Z^H(I - \Gamma^+\Gamma) = 0$ and $\Gamma^* \nabla_2 Z^H (= \Gamma^* S_Z^H \Gamma)$ is skew-Hermitian. Therefore, Z^H should be given as

$$Z^H = Z_0 - \nabla_2 Z_0 (I - \Gamma^+ \Gamma)$$

with an arbitrary matrix Z_0 and then a general skew-Hermitian solution S_Z^H to Eq. (A28) is

$$S_Z^H = \nabla_2 Z_0 \Gamma^+ - (\nabla_2 Z_0 \Gamma^+)^* + (I - \Gamma\Gamma^+) S_0 (I - \Gamma\Gamma^+) \quad \text{(A29)}$$

where S_0 is an arbitrary skew-Hermitian matrix, which can be derived by referring the results in Ref. 7. Hence, a general skew-Hermitian solution S_Z to Eq. (A18) is given as $S_Z = S_Z^S + S_Z^H$ with S_Z^S in Eq. (A26) and S_Z^H in Eq. (A29).

Next, we show the proof of part 2. The matrix $\nabla_1 S_Z \nabla_1$ satisfying Eq. (A18) should be represented as

$$\nabla_1 S_Z \nabla_1 = X^{-1}(S_Q + \Delta_1 S_L \Delta_1) X^{-1} \quad \text{(A30)}$$

where S_L is an appropriate skew-Hermitian matrix such that Eq. (A30) has a skew-Hermitian solution S_Z, which requires

$$X^{-1}(S_Q + \Delta_1 S_L \Delta_1) X^{-1} = \nabla_1 X^{-1}(S_Q + \Delta_1 S_L \Delta_1) X^{-1} \nabla_1 \quad \text{(A31)}$$

or, equivalently,

$$\nabla_2 X^{-1}(S_Q + \Delta_1 S_L \Delta_1) = 0 \quad \text{(A32)}$$

which can be derived by premultiplying both sides of Eq. (A31) by V_M^*. A necessary condition for Eq. (A31) to have a skew-Hermitian solution S_L is described in

$$(I - LL^+) \nabla_2 X^{-1} S_Q = 0$$

because $\nabla_2 X^{-1} S_Q \Delta_1 = \nabla_2 X^{-1} S_Q$, which is equivalent to condition (A22). Under this condition, a general skew-Hermitian solution S_L to Eq. (A31) can be derived in a similar way to the case of Eq. (A24), as

$$S_L = -L^+ \nabla_2 X^{-1} S_Q \Delta_1 - \Delta_1 S_Q X^{-1} \nabla_2 (L^+)^* + L^+ \nabla_2 X^{-1} S_Q X^{-1} \nabla_2 (L^+)^* + L^+ Z_0 \Delta_2 - (L^+ Z_0 \Delta_2)^* + (I - L^+ L) S_0 (I - L^+ L)$$

where Z_0 and S_0 are matrices denoted in Lemma A2. And then the solution S_Z to Eq. (A18) can be immediately given in Eq. (A23). (QED)

Appendix B: Proof of Theorem 2

Recall Eq. (A4) and suppose that every condition in Theorem 1 is satisfied. Then Eq. (A4) with $Q_{22} = S_{22} = 0$, $Q_{21} + S_{21} = 0$, and S_{11} in Eq. (A19) becomes

$$\begin{aligned} BGMX &= -\tfrac{1}{2}[\Delta_1 Q \Delta_1 + 2\Delta_1 Q \Delta_2 + \Delta_1 (S_Q - X \nabla_1 S_Z \nabla_1 X) \Delta_1] \\ &= -\tfrac{1}{2}[\Delta_1 Q \Delta_1 + 2\Delta_1 Q \Delta_2 + S_Q - X \nabla_1 S_Z \nabla_1 X] \end{aligned} \quad \text{(B1)}$$

In the case of S_Z given in Eq. (A21) of Lemma A1,

$$\begin{aligned} \nabla_1 S_Z \nabla_1 = {} & \nabla_1 [-2\bar{Q} X \Delta_2 \Gamma^+ + \bar{Q}\Gamma\Gamma^+ + 2(\Gamma^+)^* \Delta_2 X \bar{Q} + (I - \Gamma\Gamma^+) S_s (I - \Gamma\Gamma^+)] \nabla_1 - (\Gamma^+)^* \Gamma^* \bar{Q} \\ & - (\Gamma^+)^* \Delta_2 Q X \Gamma\Gamma^+ + \Gamma\Gamma^+ X^{-1} Q \Delta_2 \Gamma^+ \end{aligned} \quad \text{(B2)}$$

since

$$\nabla_1 X^{-1} S_Q \Delta_2 \Gamma^+ = -\nabla_1 X^{-1} Q (2I - X^{-1} \nabla_1 X \nabla_2) \Gamma^+$$

and

$$(\Gamma^+)^* \Delta_2 S_Q \Delta_2 \Gamma^+ = (\Gamma^+)^* \Delta_2 (Q X^{-1} \nabla_1 X - X \nabla_1 X^{-1} Q) \Delta_2 \Gamma^+$$

By substituting Eq. (B2) into Eq. (B1), we have

$$\begin{aligned} BGMX = {} & -\tfrac{1}{2}[Q - QX^{-1}\nabla_2 X + X \nabla_2 X^{-1} Q] \\ & + \tfrac{1}{2} X \nabla_1 [-\Phi + \Phi^*] \nabla_1 X \\ & + \tfrac{1}{2} X \nabla_1 (I - \Gamma\Gamma^+) S_s (I - \Gamma\Gamma^+) \nabla_1 X \end{aligned}$$

to which a general solution can be given immediately as Eq. (11) in the theorem with $\hat{S} = \tfrac{1}{2} S_s$.

A controller gain G in Eq. (13) can be derived in the similar way by using S_Z in Eq. (A23) of Lemma A1.

Appendix C: Proof of Lemma 1

The controllability of the pair $(A + BGM, D)$ means

$$\text{rank}[A + BGM - sI \quad D] = n_x + n_z, \forall s \in C$$

which can be rewritten as

$$\text{rank}[A + BGM - sI \quad D] = n_x + n_z, \forall s \in C - J \quad \text{(C1)}$$

$$\text{rank}[A + BGM - sI \quad D] = n_x + n_z, \forall s \in J \quad \text{(C2)}$$

where C denotes the set of all complex numbers and J denotes the $j\omega$ axis. We will first show that Eq. (C1) is always satisfied when Eq. (5) holds. If there exists an $s_0 \in C - J$ such that Eq. (C1) is not valid, then we can choose a vector $x (\neq 0)$ corresponding to s_0 satisfying

$$x^*(A + BGM) = s_o x^*, \qquad x^* D = 0$$

Then, from Eq. (5) we get

$$\begin{aligned} 0 &= x^* X (A + BGM)^* x + x^* (A + BGM) X x + x^* D W D^* x \\ &= 2\text{Re}(s_0) x^* X x \end{aligned}$$

which requires $\text{Re}(s_0) = 0$ because $X > 0$. This is the contradiction. Therefore, only condition (C2) is required for the controllability of the pair $(A + BGM, D)$.

By expanding Eq. (C2) as

$$\text{rank}\begin{bmatrix} A_p + B_pG_cM_p - sI_{n_x} & B_pC_c & D_p \\ B_cM_p & A_c - sI_{n_c} & 0 \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} I_{n_x} & B_pG_c & B_pC_c \\ 0 & B_c & A_c - I_{n_c} \end{bmatrix}\begin{bmatrix} A_p - sI_{n_c} & 0 & D_p \\ M_p & 0 & 0 \\ 0 & I_{n_c} & 0 \end{bmatrix} \tag{C3}$$

we can see from Sylvester's formula on matrix rank that

$$\text{rank}\begin{bmatrix} I_{n_x} & B_pG_c & B_pC_c \\ 0 & B_c & A_c - sI_{n_c} \end{bmatrix} = n_x + n_c, \qquad \forall s \in J \tag{C4}$$

and

$$\text{rank}\begin{bmatrix} A_p - sI_{n_x} & 0 & D_p \\ M_p & 0 & 0 \\ 0 & I_{n_c} & 0 \end{bmatrix} = n_x + n_z + n_c, \qquad \forall s \in J \tag{C5}$$

are sufficient conditions for Eq. (C2) to be satisfied. Equations (C4) and (C5) readily yield the conditions for Eqs. (15) and (16).

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